

# UNIVERSAL TORSORS OF DEL PEZZO SURFACES AND HOMOGENEOUS SPACES

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ABSTRACT. Let  $\text{Cox}(S_r)$  be the homogeneous coordinate ring of the blow-up  $S_r$  of  $\mathbb{P}^2$  in  $r$  general points, i.e., a smooth Del Pezzo surface of degree  $9 - r$ . We prove that for  $r \in \{6, 7\}$ ,  $\text{Proj}(\text{Cox}(S_r))$  can be embedded into  $G_r/P_r$ , where  $G_r$  is an algebraic group with root system given by the primitive Picard lattice of  $S_r$  and  $P_r \subset G_r$  is a certain maximal parabolic subgroup.

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## 1. INTRODUCTION

In this note we continue our investigations of universal torsors over Del Pezzo surfaces over an algebraically closed field  $\mathbb{K}$  of characteristic 0. The blow-up  $S_r$  of  $\mathbb{P}^2$  in  $r \leq 8$  points in *general position*<sup>1</sup> is a smooth Del Pezzo surface of degree  $9 - r$ ; we will assume that  $r \in \{3, \dots, 7\}$ . A smooth Del Pezzo surface of degree 3 (resp. degree 2) is a smooth cubic surface in  $\mathbb{P}^3$  (resp. a double cover of  $\mathbb{P}^2$  ramified in a smooth curve of degree 4). The Picard group  $\text{Pic}(S_r)$  is a lattice with a non-degenerate symmetric linear form  $(\cdot, \cdot)$ , the *intersection form*. It is well-known that  $\text{Pic}(S_r)$  contains a canonical root system  $R_r$ , which carries the action of the associated Weyl group  $W_r$ , see Table 1 and [Man86].

It was a general expectation that the Weyl group symmetry on  $\text{Pic}(S_r)$  should be a reflection of a geometric link between Del Pezzo surfaces and algebraic groups. Here we show that universal torsors of smooth Del Pezzo surfaces of degree 2 and 3 admit an embedding into a certain flag variety for the corresponding algebraic group. The degree 5 case goes back to Salberger

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<sup>1</sup>no three on a line, no six on a conic, no eight and one of them singular on a cubic curve

$r$	3	4	5	6	7
$R_r$	$\mathbf{A}_2 + \mathbf{A}_1$	$\mathbf{A}_4$	$\mathbf{D}_5$	$\mathbf{E}_6$	$\mathbf{E}_7$
$N_r$	6	10	16	27	56

TABLE 1. The root systems associated to Del Pezzo surfaces.

(talk at the Borel seminar Bern, June 1993) following Mumford [MS72], and independently Skorobogatov [Sko93]. The degree 4 case was treated in the thesis of Popov [Pop01, Chapter 6]. The existence of such an embedding in general was conjectured by Batyrev in his lecture at the conference *Diophantine geometry* (Universität Göttingen, June 2004). Skorobogatov announced related work in progress (joint with Serganova) at the conference *Cohomological approaches to rational points* (MSRI, March 2006).

As in [BP04, Section 2], the simple roots of  $R_{r-1}$  (with  $R_2 = \mathbf{A}_2$ ) can be identified with a subset  $I_r$  of the simple roots of  $R_r$  such that the edges in the Dynkin diagrams are respected. The complement of  $I_r$  in  $R_r$  consists of exactly one simple root  $\alpha_r$ , with associated fundamental weight  $\varpi_r$ . Let  $G_r$  be a simply connected linear algebraic group associated to  $R_r$ , and fix a Borel group containing a maximal torus. The Weyl group  $W_r$  acts on the weight lattice of  $G_r$ . The fundamental representation  $\varrho_r$  of  $G_r$  with highest weight  $\varpi_r$  has dimension  $N_r$  as listed in Table 1; the weights of  $\varrho_r$  can be identified with classes of curves  $E \subset S_r$  with self-intersection number  $(E, E) = -1$ , so called  $(-1)$ -curves.

Let  $P_r$  be the maximal parabolic subgroup corresponding to  $I_r$ . By [Ses78], we can regard the  $G_r$ -orbit  $H_r$  of the weight space to  $\varpi_r$  as the affine cone over  $G_r/P_r$ , and  $H_r$  is given by quadratic equations in affine space  $\mathbb{A}^{N_r}$ . For  $r = 6$ , the equations are all partial derivatives of a certain cubic form on the 27-dimensional representation  $\varrho_6$  of  $G_6$ . For  $r = 7$ , equations can be found in [Fre53]. In both cases, the equations were already known to E. Cartan in the 19th century. See Section 3 for more details on  $G_r$ ,  $\varrho_r$ , and  $H_r$  for  $r \in \{6, 7\}$ .

The *universal torsor*  $\mathcal{T}_r$  over  $S_r$  is defined as follows: Let  $\mathcal{L}_0, \dots, \mathcal{L}_r$  be a basis of  $\text{Pic}(S_r) \cong \mathbb{Z}^{r+1}$ , and let  $\mathcal{L}_i^\circ := \mathcal{L}_i \setminus \{\text{zero-section}\}$ . Then

$$\mathcal{T}_r := \mathcal{L}_0^\circ \times_{S_r} \cdots \times_{S_r} \mathcal{L}_r^\circ.$$

It is a  $T_{\text{NS}}(S_r)$ -bundle over  $S_r$ , where  $T_{\text{NS}}(S_r)$  is the Néron-Severi torus of  $S_r$ .

The *total coordinate ring*, or *Cox ring* of  $S_r$  is defined as

$$\text{Cox}(S_r) := \bigoplus_{(\nu_0, \dots, \nu_r) \in \mathbb{Z}^{r+1}} H^0(S_r, \mathcal{L}_0^{\otimes \nu_0} \otimes \cdots \otimes \mathcal{L}_r^{\otimes \nu_r})$$

as a vectorspace, and the multiplication is induced by the multiplication of sections (see [HK00] and [BP04]). It is naturally graded by  $\text{Pic}(S_r)$ , and it is generated by  $N_r$  sections corresponding to the  $(-1)$ -curves on  $S_r$ . The ideal of relations in  $\text{Cox}(S_r)$  is generated by certain quadratic relations which are homogeneous with respect to the  $\text{Pic}(S_r)$ -grading (see [BP04] for more details). Let  $\mathbb{A}(S_r) := \text{Spec}(\text{Cox}(S_r)) \subset \mathbb{A}^{N_r}$  be the corresponding affine

variety. The universal torsor  $\mathcal{T}_r$  is an open subset of  $\mathbb{A}(S_r)$  (cf. [HK00]). See [HT04] and [Der06b] for the calculation of universal torsors and Cox rings for *singular* Del Pezzo surfaces and [Der06a] for the smooth cases.

Universal torsors can be applied to *Manin's conjecture* (see [FMT89] and [BM90]) on the number of rational points of bounded height on Del Pezzo surfaces. For the moment, let the Del Pezzo surface  $S_r$  be defined over a number field  $k$ , and let  $H : S_r(k) \rightarrow \mathbb{R}$  be the anticanonical height function. Let  $U$  be the complement of the  $(-1)$ -curves on  $S_r$ . Then Manin's conjecture predicts that

$$N_{U,H}(B) := \#\{\mathbf{x} \in U(k) \mid H(\mathbf{x}) \leq B\}$$

behaves asymptotically as

$$N_{U,H}(B) \sim c \cdot B \cdot (\log B)^{n-1}$$

for some positive constant  $c$ , where  $n$  is the rank of the Picard group (over  $k$ ) of  $S_r$ .

In results concerning Manin's conjecture for various Del Pezzo surfaces (see [Bro05] and [DT06, Section 1] for an overview), often the first step is a translation of the counting problem for rational points on  $S_r$  to the counting of integral points in certain ranges on a universal torsor  $\mathcal{T}_r$ . The number of these points on  $\mathcal{T}_r$  can then be estimated using techniques from analytic number theory.

Salberger [Sal98] gave a proof of Manin's conjecture for toric varieties, which include smooth Del Pezzo surfaces of degree  $\geq 6$ , using universal torsors. De la Bretèche [Bre02] used Salberger's and Skorobogatov's description of the universal torsor as a homogeneous space in his proof of the asymptotic formula for a Del Pezzo surface of degree 5. In lower degrees, Manin's conjecture has been proved only for examples of singular Del Pezzo surfaces (see [BBD05] for a singular cubic surface, using the universal torsor), but it is a general expectation that universal torsors should lead to a proof of Manin's conjecture also in the remaining smooth cases.

We have seen that both  $\mathbb{A}(S_r)$  and  $H_r$  can be viewed as embedded into  $\mathbb{A}^{N_r}$ , with a natural identification of the coordinates. For the embedding of  $\mathbb{A}(S_r)$ , we have some freedom: As the generators of  $\text{Cox}(S_r)$  are canonical only up to a non-zero constant, we can choose a rescaling factor for each of the  $N_r$  coordinates, giving a  $N_r$ -parameter family of embeddings of  $\mathbb{A}(S_r)$  into affine space. The task is to find a rescaling such that  $\mathbb{A}(S_r)$  is embedded into  $H_r$ .

More precisely, we start with an arbitrary embedding

$$\mathbb{A}(S_r) \subset \mathbb{A}_r := \text{Spec}(\mathbb{K}[\xi(E) \mid E \text{ is a } (-1)\text{-curve on } S_r]) \cong \mathbb{A}^{N_r},$$

and view

$$H_r \subset \mathbb{A}'_r := \text{Spec}(\mathbb{K}[\xi'(E) \mid E \text{ is a } (-1)\text{-curve on } S_r]) \cong \mathbb{A}^{N_r}$$

as embedded into a different affine space. An isomorphism  $\phi_r : \mathbb{A}_r \rightarrow \mathbb{A}'_r$  such that

$$\phi_r^*(\xi'(E)) = \xi''(E) \cdot \xi(E)$$

for each of the  $N_r$  coordinates, with  $\xi''(E) \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$ , is called a *rescaling*, and the factors  $\xi''(E)$  are (a system of) *rescaling factors*. A rescaling  $\phi_r$  which embeds  $\mathbb{A}(S_r)$  into  $H_r$  is called a *good rescaling*.

Our main result is:

**Theorem 1.** *Let  $S_r$  be a smooth Del Pezzo surface of degree  $9-r$  and  $\mathbb{A}(S_r)$  the affine variety described above. Let  $H_r$  be the affine cone over the flag variety  $G_r/P_r$  associated to the root system  $R_r$  as in Table 1.*

*For  $r \in \{6, 7\}$ , there exists a  $(N_{r-2} + 2)$ -parameter family of good rescalings  $\phi_r$  which embed  $\mathbb{A}(S_r)$  into  $H_r$ .*

*Remark 2.* The number  $N_{r-2} + 2$  of parameters is 12 for  $r = 6$ , respectively 18 for  $r = 7$ .

The rescaling factors are naturally graded by  $\text{Pic}(S_r) \cong \mathbb{Z}^{r+1}$ , and we will see in Section 4 that the conditions for good rescaling are homogeneous with respect to this grading. Therefore, for each good rescaling  $\phi_r$ , there is a  $(r+1)$ -parameter family of good rescalings which differ from  $\phi_r$  only by the action of  $T_{\text{NS}}(S_r) \cong \mathbb{G}_m^{r+1}$ . Similarly,  $T_{\text{NS}}(S_r)$  acts on  $\mathbb{A}(S_r)$ , and it is easy to see that the image of  $\mathbb{A}(S_r)$  in  $H_r$  is the same for all good rescalings in the same  $(r+1)$ -parameter family. Therefore, the  $(N_{r-2} + 2)$ -parameter family of good rescalings gives rise to a  $(N_{r-2} - r + 1)$ -parameter family (where  $N_{r-2} - r + 1$  equals 5 for  $r = 6$ , resp. 10 for  $r = 7$ ) of images of  $\mathbb{A}(S_r)$  in  $H_r$ .

For  $r = 5$ , we have  $N_{r-2} - r + 1 = 2$ , and by [Pop01, Section 6.3], there is a two-parameter family of images of  $\mathbb{A}(S_5)$  under good rescalings in  $H_5$ .

In Section 2, we summarize results of [BP04] and [Der06a] on Cox rings of Del Pezzo surfaces of degree 3 and 2. In Section 3, we recall the classical equations for the homogeneous spaces  $G_r/P_r$  and give a simplified description on a certain Zariski open subset; this will help to find good rescalings. In Section 4, we derive conditions on the rescaling factors in terms of the description of  $\text{Cox}(S_r)$  and  $G_r/P_r$ . In Section 5 and Section 6, we determine good rescalings in degree 3 and 2, finishing the proof of Theorem 1.

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## 2. COX RINGS OF DEL PEZZO SURFACES

In this section, we describe the Cox ring of Del Pezzo surfaces of degree 3 and 2 in order to fix some notation. The results can be found in [BP04] and [Der06a].

Let  $r \in \{6, 7\}$ . Without loss of generality, we may assume that four of the  $r$  blown-up points in  $\mathbb{P}^2$  giving  $S_r$  are in the positions

$$p_1 = (1 : 0 : 0), \quad p_2 = (0 : 1 : 0), \quad p_3 = (0 : 0 : 1), \quad p_4 = (1 : 1 : 1).$$

By [BP04, Theorem 3.2], the generators of  $\text{Cox}(S_r)$  are sections  $\xi(E)$  vanishing in a  $(-1)$ -curve  $E$  on  $S_r$ . Their number is  $N_r$  as in Table 1. We use the same symbols  $\xi(E)$  for the coordinates in  $\mathbb{A}_r$ . Let  $-K_r$  be the anticanonical divisor class of  $S_r$ .

A  $(k)$ -ruling  $D \in \text{Pic}(S_r)$  is the sum of two  $(-1)$ -curves whose intersection number is  $k$ , cf. [BP04, Definition 4.6] and [Der06a, Definition 1]. The relations come in groups of  $r - 3$  for each  $(1)$ -ruling  $D$ . We will denote them by

$$F_{D,1}, \dots, F_{D,r-3}.$$

For  $r = 7$ , we have further relations corresponding to the  $(2)$ -ruling  $-K_7$ . The ideal  $J_r$  generated by these relations defines  $\mathbb{A}(S_r)$  (see [BP04, Theorem 4.9] for  $r \leq 6$  and [Der06a, Theorem 2] for  $r = 7$ ).

For  $r = 6$ , we assume

$$p_5 = (1 : a : b), \quad p_6 = (1 : c : d).$$

The  $N_6 = 27$   $(-1)$ -curves  $E$  are denoted by  $E_i$ ,  $m_{i,j}$ , and  $Q_i$  corresponding to the six blown-up points, the 15 transforms of the lines through two of the six points, and the six transforms of the conics through five points as described in [Der06a, Section 3]. Let  $\eta_i := \xi(E_i)$ ,  $\mu_{i,j} := \xi(m_{i,j})$ , and  $\lambda_i := \xi(Q_i)$ . We order them in the following way:

$$\eta_1, \dots, \eta_6, \quad \mu_{1,2}, \dots, \mu_{1,6}, \mu_{2,3}, \dots, \mu_{2,6}, \mu_{3,4}, \dots, \mu_{5,6}, \quad \lambda_1, \dots, \lambda_6.$$

The  $(1)$ -rulings are  $-K_6 - E$ , where  $E$  runs through the  $(-1)$ -curves. We have

$$F_{-K_6-E,1} = q_E, \quad F_{-K_6-E,2} = q'_E, \quad F_{-K_6-E,3} = q''_E,$$

where the 81 equations  $q_E, q'_E, q''_E$  are listed in [Der06a, Section 3].

For  $r = 7$ , let

$$p_5 = (1 : a_1 : b_1), \quad p_6 = (1 : a_2 : b_2), \quad p_7 = (1 : a_3 : b_3).$$

The  $N_7 = 56$   $(-1)$ -curves  $E_i, m_{i,j}, Q_i, C_i$  and the corresponding generators  $\xi(E)$  are described in [Der06a, Section 4]. They are ordered as

$$\eta_1, \dots, \eta_7, \mu_{1,2}, \dots, \mu_{1,7}, \mu_{2,3}, \dots, \mu_{6,7}, \nu_{1,2}, \dots, \nu_{1,7}, \nu_{2,3}, \dots, \nu_{6,7}, \lambda_1, \dots, \lambda_7.$$

By [Der06a, Theorem 2], there are 529 relations  $F_{D,i}$  (four for each of the 126  $(1)$ -rulings, and 25 for the  $(2)$ -ruling  $-K_7$ ) between them. We do not want to list them here, as they can be determined by the method of [Der06a, Lemma 4] in a straightforward manner.

## 3. HOMOGENEOUS SPACES

In this section, we examine the equations defining the affine cone  $H_r \subset \mathbb{A}'_r$  over  $G_r/P_r$  for  $r \in \{6, 7\}$ . For the  $N_r$  coordinates  $\xi'(E)$  of  $\mathbb{A}'_r$ , we also use the names  $\eta'_i$ ,  $\mu'_{i,j}$ ,  $\lambda'_i$ , and furthermore  $\nu'_{i,j}$  in the case  $r = 7$ , with the obvious correspondence to the coordinates of  $\mathbb{A}_r$  as in the previous section.

In particular, we show that  $H_r$  is a complete intersection on the open subset  $U_r$  of  $\mathbb{A}'_r$  where the coordinates  $\eta'_1, \dots, \eta'_r$  are non-zero.

We will see that  $H_r$  is defined by quadratic relations which are homogeneous with respect to the  $\text{Pic}(S_r)$ -grading. For each (1)-ruling  $D$ , we have exactly one relation  $p_D$  of degree  $D$ , and furthermore in the case  $r = 7$ , we have eight relations  $p_{-K_7}^{(1)}, \dots, p_{-K_7}^{(8)}$  where we use the convention  $p_{-K_7} := p_{-K_7}^{(1)}$ . For any possibility to write  $D$  as the sum of two  $(-1)$ -curves  $E, E'$ , the relation  $p_D$  has a term  $\xi'(E)\xi'(E')$  with a non-zero coefficient.

**Definition 3.** For a  $(-1)$ -curve  $E$ , let  $U_E$  be the open subset of  $\mathbb{A}'_r$  where  $\xi'(E)$  is non-zero. Let  $\mathcal{N}(E)_k$  be the set of  $(-1)$ -curves  $E'$  with  $(E, E') = k$ , and let  $\Xi'(E)_k$  be the set of the corresponding  $\xi'(E')$ . Let  $\mathcal{N}(E)_{>k}$  and  $\Xi'(E)_{>k}$  be defined similarly, but with the condition  $(E, E') > k$ .

Let  $U_r \subset \mathbb{A}'_r$  be the intersection of  $U_{E_1}, \dots, U_{E_r}$ .

Note that  $\mathcal{N}(E)_0$  has exactly  $N_{r-1}$  elements because we can identify its elements with the  $(-1)$ -curves on  $S_{r-1}$ . Since the only  $(-1)$ -curve intersecting  $E$  negatively is  $E$  itself, the number of elements of  $\mathcal{N}(E)_{>0}$  is  $N_r - N_{r-1} - 1$ .

**Proposition 4.** *Let*

$$\Phi_r : H_r \cap U_r \rightarrow U_{r-1} \times (\mathbb{A}^1 \setminus \{0\})$$

*be the projection to the coordinates  $\xi'(E) \in \Xi'(E_1)_0$  and  $\eta'_1$ . The map  $\Phi_r$  is an isomorphism. The dimension of  $H_r$  is  $N_{r-1} + 1$ .*

*Proof.* If  $D = E_1 + E$  is a (1)-ruling, then  $(E_1, D) = 0$ , and all variables occurring in  $p_D$  besides  $\eta'_1$  and  $\xi'(E)$  are elements of  $\Xi'(E_1)_0$ . For  $\eta'_1 \neq 0$ , the relation  $p_D$  expresses  $\xi'(E)$  in terms of  $\eta'_1$  and  $\Xi'(E_1)_0$ .

For a (2)-ruling  $D = E_1 + E$ , we have  $(E_1, D) = 1$ , so the relation  $p_D$  expresses  $\xi'(E)$  in terms of  $\eta'_1$  and monomials  $\xi'(E'_i)\xi'(E''_i)$  where  $\xi(E'_i) \in \Xi'(E_1)_0$  and  $\xi(E''_i) \in \Xi'(E_1)_1$ . Using the expressions for the elements of  $\Xi'(E_1)_1$  of the first step, this shows that we can express the coordinates  $\Xi'(E_1)_{>0}$  in terms of  $\eta'_1$  and  $\Xi'(E_1)_0$  by using the  $N_r - N_{r-1} - 1$  relations  $g_{E_1+E}$  for  $E \in \mathcal{N}(E)_{>0}$ . This allows us to construct a map

$$\Psi_r : U_{r-1} \times (\mathbb{A}^1 \setminus \{0\}) \rightarrow \mathbb{A}'_r.$$

It remains to show that the image of  $\Psi_r$  is in  $H_r$ , i.e., that the resulting point also satisfies the remaining equations which define  $H_r$ . This is done in Lemma 6 and Lemma 7 below.  $\square$

*Remark 5.* Proposition 4 is also true if we enlarge  $U_r$  to  $U_{E_1}$  and  $U_{r-1}$  to  $\mathbb{A}'_{r-1}$ . However, the proofs of Lemma 6 and Lemma 7 are slightly simplified by restricting to  $U_r$ .

First, we consider the case  $r = 6$ . Consider the cubic form in  $N_6 = 27$  variables

$$F(M_1, M_2, M_3) := \det M_1 + \det M_2 + \det M_3 - \operatorname{tr}(M_1 M_2 M_3),$$

where

$$M_1 := \begin{pmatrix} \eta'_1 & \lambda'_1 & \mu'_{2,3} \\ \eta'_2 & \lambda'_2 & \mu'_{1,3} \\ \eta'_3 & \lambda'_3 & \mu'_{1,2} \end{pmatrix}, \quad M_2 := \begin{pmatrix} \lambda'_4 & \lambda'_5 & \lambda'_6 \\ \eta'_4 & \eta'_5 & \eta'_6 \\ \mu'_{5,6} & \mu'_{4,6} & \mu'_{4,5} \end{pmatrix},$$

and

$$M_3 := \begin{pmatrix} \mu'_{1,4} & \mu'_{2,4} & \mu'_{3,4} \\ \mu'_{1,5} & \mu'_{2,5} & \mu'_{3,5} \\ \mu'_{1,6} & \mu'_{2,6} & \mu'_{3,6} \end{pmatrix}.$$

By [Spr06, Proposition 1.6], the group of invertible  $N_6 \times N_6$ -matrices which leave invariant  $F$  is a simply connected linear algebraic group  $G_6$  of type  $\mathbf{E}_6$ .

Note that the terms of  $\operatorname{tr}(M_1 M_2 M_3)$  are  $M_1^{(i,j)} M_2^{(j,k)} M_3^{(k,i)}$  for  $i, j, k \in \{1, 2, 3\}$  (where  $M_a^{(b,c)}$  is the entry  $(b, c)$  of the matrix  $M_a$ ), so the number of terms of  $F$  is  $3 \cdot 6 + 3^3 = 45$ . Each is a product of three variables  $\xi'(E)$ ,  $\xi'(E')$ ,  $\xi'(E'')$  such that the corresponding  $(-1)$ -curves  $E, E', E''$  on  $S_6$  form a triangle, and their divisor classes add up to  $-K_6$ . The coefficient is  $+1$  in the nine cases

$$\begin{array}{lll} \eta'_1 \mu'_{1,2} \lambda'_2, & \eta'_2 \mu'_{2,3} \lambda'_3, & \eta'_3 \mu'_{1,3} \lambda'_1, \\ \eta'_4 \mu'_{4,6} \lambda'_6, & \eta'_5 \mu'_{4,5} \lambda'_4, & \eta'_6 \mu'_{5,6} \lambda'_5, \\ \mu'_{1,4} \mu'_{2,5} \mu'_{3,6}, & \mu'_{1,5} \mu'_{2,6} \mu'_{3,4}, & \mu'_{1,6} \mu'_{2,4} \mu'_{3,5} \end{array}$$

and  $-1$  in the remaining 36 cases. (Of course, there is some choice here, for example by permuting the indices  $1, \dots, 6$ , but it is not as simple as choosing any 9 of the 45 terms to have the coefficient  $+1$ . See [Lur01, Section 5] for more details.)

Let  $\alpha_6$  be the simple root at the end of one of the “long legs” in the Dynkin diagram  $\mathbf{E}_6$ . Let  $\varpi_6$  be the associated fundamental weight. The action of  $G_6$  on  $\mathbb{K}^{N_6}$  is a  $N_6$ -dimensional irreducible representation of  $G_6$  whose highest weight is  $\varpi_6$  (cf. [Che58, Section 20.2]). The orbit  $H_6$  of the weight space of  $\varpi_6$  is described by the vanishing of the  $N_6$  partial derivatives of the cubic form  $F$  (see [Zak93, Section III.2.5]).

The derivative with respect to  $\xi'(E)$  contains five terms  $\pm \xi'(E') \xi'(E'')$  corresponding to the five ways to write the  $(1)$ -ruling  $D := -K_6 - E$  as the sum of two intersecting  $(-1)$ -curves  $E', E''$ . We will denote it by  $p_D = p_{-K_6-E}$ .

**Lemma 6.** *For  $\eta'_1 \neq 0$  and any values of*

$$\Xi'(E_1)_0 = \{\eta'_2, \dots, \eta'_6, \mu'_{2,3}, \dots, \mu'_{5,6}, \lambda'_1\}$$

*with non-zero  $\eta'_2, \dots, \eta'_6$ , the equations  $p_{E_1+E}$  for*

$$E \in \mathcal{N}(E_1)_1 = \{m_{1,2}, \dots, m_{1,6}, Q_2, \dots, Q_6\}$$

*define a point of  $H_6$ .*

*Proof.* As  $T_{\text{NS}}(S_6)$  acts on  $H_6$  and  $\{E_1, \dots, E_6\}$  is a subset of a basis of  $\text{Pic}(S_6)$ , we may assume that  $\eta'_1 = \dots = \eta'_6 = 1$ . Then for  $i \in \{2, \dots, 6\}$ , the equation  $p_{E_1+m_{1,i}}$  allows us to express  $\mu'_{1,i}$  in terms of the remaining  $\mu'_{i,j}$ :

$$\begin{aligned}\mu'_{1,2} &= \mu'_{2,3} + \mu'_{2,4} + \mu'_{2,5} + \mu'_{2,6}, & \mu'_{1,3} &= \mu'_{2,3} - \mu'_{3,4} - \mu'_{3,5} - \mu'_{3,6}, \\ \mu'_{1,4} &= -\mu'_{2,4} - \mu'_{3,4} + \mu'_{4,5} - \mu'_{4,6}, & \mu'_{1,5} &= -\mu'_{2,5} - \mu'_{3,5} - \mu'_{4,5} + \mu'_{5,6}, \\ \mu'_{1,6} &= -\mu'_{2,6} - \mu'_{3,6} + \mu'_{4,6} - \mu'_{5,6}\end{aligned}$$

Furthermore, for  $i \in \{2, \dots, 6\}$ , we can use  $p_{E_1+Q_i}$  in order to express  $\lambda'_i$  in terms of  $\lambda'_1$  and  $\mu'_{j,k}$ :

$$\begin{aligned}\lambda'_2 &= \mu'_{3,4}\mu'_{5,6} + \mu'_{3,5}\mu'_{4,6} + \mu'_{3,6}\mu'_{4,5} + \lambda'_1 \\ \lambda'_3 &= -\mu'_{2,4}\mu'_{5,6} - \mu'_{2,5}\mu'_{4,6} - \mu'_{2,6}\mu'_{4,5} + \lambda'_1 \\ \lambda'_4 &= -\mu'_{2,3}\mu'_{5,6} + \mu'_{2,5}\mu'_{3,6} - \mu'_{2,6}\mu'_{3,5} - \lambda'_1 \\ \lambda'_5 &= -\mu'_{2,3}\mu'_{4,6} - \mu'_{2,4}\mu'_{3,6} + \mu'_{2,6}\mu'_{3,4} - \lambda'_1 \\ \lambda'_6 &= -\mu'_{2,3}\mu'_{4,5} + \mu'_{2,4}\mu'_{3,5} - \mu'_{2,5}\mu'_{3,4} - \lambda'_1\end{aligned}$$

By substituting and expanding, we check that the remaining 17 relations are fulfilled. Therefore, the resulting point lies in  $H_6$ .  $\square$

Next, we obtain similar results in the case  $r = 7$  with  $N_7 = 56$ . By [Spr06, Corollary 2.6], a simply connected linear algebraic group  $G_7$  of type  $\mathbf{E}_7$  is obtained as the identity component of the group of invertible  $N_7 \times N_7$ -matrices which leave invariant a certain quartic form defined on a vectorspace of dimension  $N_7$  as in [Spr06, Section 2.1]. The action of  $G_7$  on this vectorspace is an irreducible representation whose highest weight  $\varpi_7$  is the fundamental weight corresponding to the simple root  $\alpha_7$  at the end of the “longest leg” of the Dynkin diagram  $\mathbf{E}_7$  (cf. [Che58, Section 20.2]).

We describe the orbit  $H_7$  of the weight space of  $\varpi_7$  under  $G_7$  explicitly. The  $N_7$  coordinates  $\xi'(E)$  in  $\mathbb{A}'_7$  are  $\eta'_i, \mu'_{j,k}, \nu'_{j,k}, \lambda'_i$  for  $i, j, k \in \{1, \dots, 7\}$  and  $j < k$ . The equations for  $H_7$  are described in [Fre53] in terms of 56 coordinates  $x^{ij}, y_{ij}$  ( $i < j \in \{1, \dots, 8\}$ ). They correspond to our variables as follows:

$$\eta'_i = x^{i8}, \quad \mu'_{k,l} = y_{kl}, \quad \nu'_{k,l} = x^{kl}, \quad \lambda'_i = y_{i8}.$$

For the (1)-rulings  $D$  [Der06a, Lemma 11], the relations  $p_D$  are  $u^{ijkl}$  and  $v_j^i$  as below. In the first column of Table 2, we list a symbol  $D_I^{(n)}$  assigned to the (1)-ruling in the second column, and the third column gives the corresponding relation.

Let

$$u^{ijkl} = x^{ij}x^{kl} - x^{ik}x^{jl} + x^{il}x^{jk} + \sigma \cdot (y_{ab}y_{cd} - y_{ac}y_{bd} + y_{ad}y_{bc}),$$

where  $i < j < k < l$  and  $a < b < c < d$ , with  $(i, j, k, l, a, b, c, d)$  a permutation of  $(1, \dots, 8)$ , and  $\sigma$  its sign. For  $i \neq j$

$$v_j^i = \sum_{k \in (\{1, \dots, 8\} \setminus \{i, j\})} x^{ik} y_{kj},$$

where  $x^{ba} = -x^{ab}$  and  $y_{ba} = -y_{ab}$  if  $b > a$ .



symbol	(1)-ruling $D = D_I^{(n)}$	relation $p_D$
$D_i^{(1)}$	$H - E_i$	$v_i^8$
$D_{i,j,k}^{(2)}$	$2H - (E_1 + \cdots + E_7) + E_i + E_j + E_k$	$u^{ijk8}$
$D_{i,j}^{(3)}$	$3H - (E_1 + \cdots + E_7) + E_i - E_j$	$v_j^i$
$D_{i,j,k,l}^{(4)}$	$4H - 2(E_1 + \cdots + E_7) + E_i + E_j + E_k + E_l$	$u^{ijkl}$
$D_i^{(5)}$	$5H - 2(E_1 + \cdots + E_7) + E_i$	$v_8^i$

TABLE 2. Rulings and relations defining  $G_7/P_7$ .

For the (2)-ruling  $-K_7$ , we have the following eight equations with 28 terms:

$$p_{-K_7}^{(i)} := v_i^i := -\frac{3}{4} \sum_{j \in (\{1, \dots, 8\} \setminus \{i\})} x^{ij} y_{ij} + \frac{1}{4} \sum_{j < k \in (\{1, \dots, 8\} \setminus \{i\})} x^{jk} y_{jk}$$

**Lemma 7.** For  $\eta'_1, \dots, \eta'_7 \neq 0$ , the 28 coordinates

$$\eta'_i \quad (i \in \{1, \dots, 7\}), \quad \mu'_{j,k} \quad (j < k \in \{2, \dots, 7\}), \quad \nu'_{1,l} \quad (l \in \{2, \dots, 7\})$$

in  $\Xi'(E_1)_0$  and the 28 equations  $p_D$  for

$$D \in \{D_2^{(1)}, \dots, D_7^{(1)}, D_{1,2,3}^{(2)}, \dots, D_{1,6,7}^{(2)}, D_{1,2}^{(3)}, \dots, D_{1,7}^{(3)}, -K_7\}$$

define

$$\mu'_{1,i} \quad (i \in \{2, \dots, 7\}), \quad \nu'_{j,k} \quad (j < k \in \{2, \dots, 7\}), \quad \lambda'_l \quad (l \in \{1, \dots, 7\}),$$

resulting in a point on  $H_7$ .

Furthermore, we may replace  $p_{-K}$  by  $p_D$  for  $D = D_{2,1}^{(3)}$ .

*Proof.* As above, we may assume that  $\eta'_1 = \cdots = \eta'_7 = 1$  because of the action of  $T_{\text{NS}}(S_7)$ . For the 27  $(-1)$ -curves  $E \in \mathcal{N}(E_1)_1$ , the equation  $p_{E_1+E}$  defines  $\xi'(E)$  directly in terms of the 28 variables in  $\Xi'(E_1)_0$ ; we do not list the expressions here. By substituting these results, we use  $v_1^1$  in order to express  $\lambda'_1$  in terms of these variables:

$$\begin{aligned} \lambda'_1 = & -\mu'_{2,3}\mu'_{4,5}\mu'_{6,7} + \mu'_{2,3}\mu'_{4,6}\mu'_{5,7} - \mu'_{2,3}\mu'_{4,7}\mu'_{5,6} + \mu'_{2,4}\mu'_{3,5}\mu'_{6,7} - \mu'_{2,4}\mu'_{3,6}\mu'_{5,7} \\ & + \mu'_{2,4}\mu'_{3,7}\mu'_{5,6} - \mu'_{2,5}\mu'_{3,4}\mu'_{6,7} + \mu'_{2,5}\mu'_{3,6}\mu'_{4,7} - \mu'_{2,5}\mu'_{3,7}\mu'_{4,6} + \mu'_{2,6}\mu'_{3,4}\mu'_{5,7} \\ & - \mu'_{2,6}\mu'_{3,5}\mu'_{4,7} + \mu'_{2,6}\mu'_{3,7}\mu'_{4,5} - \mu'_{2,7}\mu'_{3,4}\mu'_{5,6} + \mu'_{2,7}\mu'_{3,5}\mu'_{4,6} - \mu'_{2,7}\mu'_{3,6}\mu'_{4,5} \\ & - \mu'_{2,3}\lambda'_2 + \mu'_{2,3}\lambda'_3 - \mu'_{2,4}\lambda'_2 + \mu'_{2,4}\lambda'_4 - \mu'_{2,5}\lambda'_2 + \mu'_{2,5}\lambda'_5 \\ & - \mu'_{2,6}\lambda'_2 + \mu'_{2,6}\lambda'_6 - \mu'_{2,7}\lambda'_2 + \mu'_{2,7}\lambda'_7 - \mu'_{3,4}\lambda'_3 + \mu'_{3,4}\lambda'_4 \\ & - \mu'_{3,5}\lambda'_3 + \mu'_{3,5}\lambda'_5 - \mu'_{3,6}\lambda'_3 + \mu'_{3,6}\lambda'_6 - \mu'_{3,7}\lambda'_3 + \mu'_{3,7}\lambda'_7 \\ & - \mu'_{4,5}\lambda'_4 + \mu'_{4,5}\lambda'_5 - \mu'_{4,6}\lambda'_4 + \mu'_{4,6}\lambda'_6 - \mu'_{4,7}\lambda'_4 + \mu'_{4,7}\lambda'_7 \\ & - \mu'_{5,6}\lambda'_5 + \mu'_{5,6}\lambda'_6 - \mu'_{5,7}\lambda'_5 + \mu'_{5,7}\lambda'_7 - \mu'_{6,7}\lambda'_6 + \mu'_{6,7}\lambda'_7 \end{aligned}$$

We check directly by substituting and expanding that the remaining equations defining  $H_7$  are fulfilled.

As  $v_1^2$  contains the term  $\eta'_2\lambda'_1$ , and  $\eta'_2 \neq 0$ , we may replace  $v_1^1$  by  $v_1^2$ .  $\square$

## 4. RESCALINGS

Let  $r \in \{6, 7\}$ . We follow the strategy of the case  $r = 5$  [Pop01, Section 6.3] in order to describe conditions for good rescalings explicitly in terms of the rescaling factors. However, we use the results of the previous section to simplify this as follows:

Let

$$\mathcal{M}_6 := \{E_1 + E \mid E \in \mathcal{N}(E_1)_1\}$$

and let

$$\mathcal{M}_7 := \{E_1 + E \mid E \in \mathcal{N}(E_1)_1\} \cup \{D_{2,1}^{(3)}\}.$$

Let  $\tilde{H}_r \subset \mathbb{A}'_r$  be the variety defined by the equations  $g_D$  for  $D \in \mathcal{M}_r$ .

By Proposition 4, Lemma 6, and Lemma 7,  $H_r \cap U_r = \tilde{H}_r \cap U_r$ .

*Remark 8.* Because of  $\mathcal{N}(E_1)_2 = \{C_1\}$ , it could be considered more natural to use  $-K_7 = E_1 + C_1$  instead of  $D_{2,1}^{(3)} = E_2 + C_1$  in the definition of  $\mathcal{M}_7$ . However, we choose to avoid the (2)-ruling  $-K_7$  for technical reasons.

**Lemma 9.** *A rescaling  $\phi_r : \mathbb{A}_r \rightarrow \mathbb{A}'_r$  is good if and only if it embeds  $\mathbb{A}(S_r)$  into  $\tilde{H}_r$ .*

*Proof.* As  $H_r \subset \tilde{H}_r$ , a good rescaling  $\phi_r$  satisfies  $\phi_r(\mathbb{A}(S_r)) \subset \tilde{H}_r$ . Conversely, we have

$$\phi_r(\mathbb{A}(S_r)) \cap U_r \subset \tilde{H}_r \cap U_r = H_r \cap U_r$$

by Lemma 6 and Lemma 7. Taking the closure and using that  $H_r$  is closed and that  $\mathbb{A}(S_r)$  is irreducible by [BP04], we conclude that  $\phi_r(\mathbb{A}(S_r)) \subset H_r$ , so the rescaling is good.  $\square$

As in Section 2, let  $J_r$  be the ideal defining  $\mathbb{A}(S_r)$  in  $\mathbb{A}_r$ .

In terms of the coordinate rings  $\mathbb{K}[\mathbb{A}_r]$  and  $\mathbb{K}[\mathbb{A}'_r]$  and in view of the previous lemma, a rescaling  $\phi_r$  is good if, for all  $D \in \mathcal{M}_r$ , the ideal  $J_r \subset \text{rad}(J_r)$  contains  $\phi_r^*(p_D)$ , where  $p_D$  is the equation defining  $H_r$  corresponding to the (1)-ruling  $D$ .

As  $\mathbb{K}[\mathbb{A}_r]$  and  $\mathbb{K}[\mathbb{A}'_r]$  are both graded by  $\text{Pic}(S_r)$  and  $\phi_r^*$  respects this grading, we need rescaling factors such that  $\phi_r^*(p_D)$  of degree  $D \in \mathcal{M}_r$  is a linear combination of the equations  $F_{D,1}, \dots, F_{D,r-3} \in J_r$ . For concrete calculations in the next sections, we describe this more explicitly:

Let  $D \in \mathcal{M}_r$  be a (1)-ruling, which can be written in  $r - 1$  ways as the sum of two  $(-1)$ -curves  $E'_i, E''_i$ . For  $i \in \{1, \dots, r - 1\}$ , let

$$\xi_i := \xi(E'_i)\xi(E''_i), \quad \xi'_i := \xi'(E'_i)\xi'(E''_i), \quad \xi''_i := \xi''(E'_i)\xi''(E''_i).$$

Then  $p_D$  has the form

$$(1) \quad p_D = \sum_{i=1}^{r-1} \epsilon_i \xi'_i$$

with  $\epsilon_i \in \{\pm 1\}$ .

As  $\xi_i$  vanishes exactly on  $E'_i \cup E''_i$ , the 2-dimensional space  $H^0(S_r, \mathcal{O}(D))$  is generated by any two  $\xi_i, \xi_{i'}$ . Hence, all other  $r - 3$  elements  $\xi_j$  are linear combinations of  $\xi_i, \xi_{i'}$ , with non-vanishing coefficients. This gives  $r - 3$  relations of degree  $D$  in  $\text{Cox}(S_r)$ . Rearranging  $\xi_1, \dots, \xi_{r-1}$  such that the

two elements  $\xi_i, \xi_{i'}$  of our choice have the indices  $r-2$  and  $r-1$ , we can write them as

$$(2) \quad F_{D,j} = \xi_j + \alpha_j \xi_{r-2} + \beta_j \xi_{r-1},$$

for  $j \in \{1, \dots, r-3\}$ , where  $\alpha_j, \beta_j \in \mathbb{K}^*$ .

Suppose that  $\phi_r^*(p_D)$  is a linear combination of the  $F_{D,j}$  with factors  $\lambda_j$ :

$$\phi_r^*(p_D) - \sum_{j=1}^{r-3} \lambda_j F_{D,j} = 0.$$

Since  $\phi_r^*(\xi'(E)) = \xi''(E) \cdot \xi(E)$ , we have  $\phi_r^*(\xi'_i) = \xi''_i \cdot \xi_i$  for the monomials of degree 2. Then the above equation is equivalent to the vanishing of

$$\sum_{i=1}^{r-3} (\epsilon_i \xi''_i - \lambda_i) \xi_i + \left( \epsilon_{r-2} \xi''_{r-2} - \sum_{j=1}^{r-3} \lambda_j \alpha_j \right) \xi_{r-2} + \left( \epsilon_{r-1} \xi''_{r-1} - \sum_{j=1}^{r-3} \lambda_j \beta_j \right) \xi_{r-1}.$$

For  $i \in \{1, \dots, r-3\}$ , we see by considering the coefficients of  $\xi_i$  that we must choose  $\lambda_i = \epsilon_i \xi''_i$ . With this, consideration of the coefficients of  $\xi_{r-2}$  and  $\xi_{r-1}$  results in the following conditions  $g_{D,1}, g_{D,2}$  on the rescaling factors  $\xi''_j$ , which are homogeneous of degree  $D \in \text{Pic}(S_r)$ :

$$g_{D,1} := \epsilon_{r-2} \xi''_{r-2} - \sum_{j=1}^{r-3} \epsilon_j \alpha_j \xi''_j = 0, \quad g_{D,2} := \epsilon_{r-1} \xi''_{r-1} - \sum_{j=1}^{r-3} \epsilon_j \beta_j \xi''_j = 0.$$

Note that our choice of  $\xi_{r-2}$  and  $\xi_{r-1}$  in the definition of  $F_{D,j}$  as discussed before (2) is reflected here in the sense that  $g_{D,1}$  and  $g_{D,2}$  express the corresponding  $\xi''_{r-2}$  and  $\xi''_{r-1}$  as linear combinations of  $\xi''_1, \dots, \xi''_{r-3}$  with non-zero coefficients.

This information can be summarized as follows:

**Lemma 10.** *For  $r \in \{6, 7\}$ , a rescaling is good if and only if the rescaling factors  $\xi''(E)$  fulfill the equations  $g_{D,1}$  and  $g_{D,2}$  for each (1)-ruling  $D \in \mathcal{M}_r$ .*

*As described above precisely, the non-zero coefficients  $\epsilon_i$  are taken from the equations  $p_D$  (1) defining  $H_r$ , and the non-zero  $\alpha_j, \beta_j$  are taken from the equations  $F_{D,j}$  (2) defining  $\mathbb{A}(S_r)$ .*

Let  $\Xi''(E)_k$  (resp.  $\Xi''(E)_{>k}$ ) be the set of all  $\xi''(E')$  for  $E' \in \mathcal{N}(E)_k$  (resp.  $E' \in \mathcal{N}(E)_{>k}$ ). Let

$$\Xi''_{i,j} := \Xi''(E_1)_i \cap \Xi''(E_2)_j.$$

We claim that we may express the rescaling factors  $\Xi''(E_1)_{>0} \cup \Xi''(E_2)_{>0}$  in terms of the other  $N_{r-2} + 2$  rescaling factors  $\{\eta''_1, \eta''_2\} \cup \Xi''_{0,0}$ .

We will prove this for  $r \in \{6, 7\}$  as follows: The  $2 \cdot (N_r - N_{r-1} - 1)$  equations  $g_{D,i}$  are homogeneous of degree  $D$  with respect to the  $\text{Pic}(S_r)$ -grading of the variables  $\xi''(E)$ , and we are interested only in the solutions where all  $\xi''(E)$  are non-zero. Because of the action of  $T_{\text{NS}}(S_r)$  on the rescaling factors and as  $E_1, \dots, E_r$  are part of a basis of  $\text{Pic}(S_r)$ , we may assume  $\eta''_1 = \dots = \eta''_r = 1$ .

Consider a (1)-ruling  $D = E_1 + E$  such that  $(E_2, E) = 0$ . Then

$$D = E'_1 + E''_1 = \dots = E'_{r-3} + E''_{r-3} = E_1 + E = E_2 + E'$$

are the  $r-1$  possibilities to write  $D$  as the sum of two intersecting  $(-1)$ -curves. Here,  $E'_i, E''_i \in \mathcal{N}(E_1)_0 \cap \mathcal{N}(E_2)_0$ . As in the discussion before

Lemma 10, we may set up the equations  $F_{D,j}$  such that  $g_{D,1}$  and  $g_{D,2}$  express  $\xi''(E)$  and  $\xi''(E')$  directly as a linear combination of  $\xi''(E'_i)\xi''(E''_i)$ . This expresses all  $\xi(E) \in \Xi''_{1,0}$  and all  $\xi''(E') \in \Xi''_{0,1}$  in terms of variables in  $\Xi''_{0,0}$ .

For a (1)-ruling  $D = E_1 + E$  such that  $(E_2, E) = 1$ , we have

$$D = E'_1 + E''_1 = \cdots = E'_{r-2} + E''_{r-2} = E_1 + E,$$

where we may assume  $(E_2, E'_i) = 0$  and  $(E_2, E''_i) = 1$ . Since  $(E_1, E'_i) = (E_1, E''_i) = 0$ , we have  $\xi''(E'_i) \in \Xi''_{0,0}$  and  $\xi''(E''_i) \in \Xi''_{0,1}$ . Using the previous findings to express  $\xi''(E''_i)$  in terms of variables  $\Xi''_{0,0}$ , the equation  $g_{D,1}$  results in a condition on the variables  $\Xi''_{0,0}$ , while  $g_{D,2}$  expresses  $\xi''(E) \in \Xi''_{1,1}$  in terms of these variables.

In the case  $r = 7$ , the equation  $g_{D,2}$  for  $D = E_1 + C_2$  expresses  $\lambda''_2 \in \Xi''_{1,2}$  in terms of variables in  $\Xi''_{0,1}$ , and  $g_{D,1}$  gives a further condition on these variables. Furthermore,  $g_{D,2}$  for the (1)-ruling  $D = E_2 + C_1$  expresses  $\lambda''_1 \in \Xi''_{2,1}$  in terms of  $\Xi''_{1,0}$ , while  $g_{D,1}$  gives a further condition on them. Substituting the expressions for  $\Xi''_{0,1}$  respectively  $\Xi''_{1,0}$  in terms of  $\Xi''_{0,0}$ , we get expressions for  $\lambda''_2$  and  $\lambda''_1$ , while the  $g_{D,1}$  result in further condition on  $\Xi''_{0,0}$ .

We summarize this in the following lemma. For its proof, it remains to show in the following sections that the expressions for  $\Xi''(E_1)_{>0} \cup \Xi''(E_2)_{>0}$  are non-zero, and that the further conditions vanish.

**Lemma 11.** *We can write the  $N_r - N_{r-2} - 2$  rescaling factors in the set  $\Xi''(E_1)_{>0} \cup \Xi''(E_2)_{>0}$  as non-zero expressions in terms of  $N_{r-2} + 2$  rescaling factors  $\Xi''_{0,0} \cup \{\xi''(E_1), \xi''(E_2)\}$ . With this, the  $N_r - 2N_{r-1} + N_{r-2}$  further conditions on the rescaling factors are trivial.*

For an open subset of the  $N_{r-2} + 2$  parameters  $\{\eta''_1, \eta''_2\} \cup \Xi''_{0,0}$ , all rescaling factors are non-zero, so we obtain good rescalings, which proves Theorem 1 once the proof of Lemma 11 is completed.

## 5. DEGREE 3

In this section, we prove Lemma 11 for  $r = 6$  by solving the system of equations on the rescaling factors of Lemma 10: For each (1)-ruling  $D \in \mathcal{M}_6$ , we determine the coefficients of the equation  $p_D$  defining  $H_6$  as in (1), and find the coefficients  $\alpha_j, \beta_j$  of  $F_{D,j}$  defining  $\mathbb{A}(S_6)$  as in (2) in the list in [Der06a, Section 3]. This allows us to write down the 20 equations  $g_{D,i}$  on the rescaling factors  $\xi''(E)$  explicitly. Let

$$\gamma_1 := ad - bc, \quad \gamma_2 := (a-1)(d-1) - (b-1)(c-1)$$

for simplicity.

$$\begin{aligned} g_{E_1+m_{1,2},1} &= -\eta''_3\mu''_{2,3} - \eta''_4\mu''_{2,4} - b\eta''_5\mu''_{2,5} - d\eta''_6\mu''_{2,6}, \\ g_{E_1+m_{1,2},2} &= \eta''_1\mu''_{1,2} + \eta''_4\mu''_{2,4} + \eta''_5\mu''_{2,5} + \eta''_6\mu''_{2,6}, \\ g_{E_1+m_{1,3},1} &= -\eta''_1\mu''_{1,3} + \eta''_4\mu''_{3,4} + \eta''_5\mu''_{3,5} + \eta''_6\mu''_{3,6}, \\ g_{E_1+m_{1,3},2} &= \eta''_2\mu''_{2,3} + \eta''_4\mu''_{3,4} + a\eta''_5\mu''_{3,5} + c\eta''_6\mu''_{3,6}, \\ g_{E_1+m_{1,4},1} &= -\eta''_1\mu''_{1,4} + \eta''_3\mu''_{3,4} + (b-1)\eta''_5\mu''_{4,5} + (1-d)\eta''_6\mu''_{4,6}, \\ g_{E_1+m_{1,4},2} &= -\eta''_2\mu''_{2,4} + \eta''_3\mu''_{3,4} + (b-a)\eta''_5\mu''_{4,5} + (c-d)\eta''_6\mu''_{4,6}, \end{aligned}$$

$$\begin{aligned}
g_{E_1+m_{1,5},1} &= -\eta_2''\mu_{2,5}'' + a/b\eta_3''\mu_{3,5}'' + (a-b)/b\eta_4''\mu_{4,5}'' + \gamma_1/b\eta_6''\mu_{5,6}'', \\
g_{E_1+m_{1,5},2} &= -\eta_1''\mu_{1,5}'' + 1/b\eta_3''\mu_{3,5}'' + (1-b)/b\eta_4''\mu_{4,5}'' + (d-b)/b\eta_6''\mu_{5,6}'', \\
g_{E_1+m_{1,6},1} &= -\eta_1''\mu_{1,6}'' + 1/d\eta_3''\mu_{3,6}'' + (d-1)/d\eta_4''\mu_{4,6}'' + (b-d)/d\eta_5''\mu_{5,6}'', \\
g_{E_1+m_{1,6},2} &= -\eta_2''\mu_{2,6}'' + c/d\eta_3''\mu_{3,6}'' + (d-c)/d\eta_4''\mu_{4,6}'' - \gamma_1/d\eta_5''\mu_{5,6}'', \\
g_{E_1+Q_{2,1}} &= a(c-d)\eta_1''\lambda_2'' + (d-1)\eta_2''\lambda_1'' - \mu_{3,4}''\mu_{5,6}'' + \mu_{3,6}''\mu_{4,5}'', \\
g_{E_1+Q_{2,2}} &= \gamma_1\eta_1''\lambda_2'' + (b-d)\eta_2''\lambda_1'' - \mu_{3,5}''\mu_{4,6}'' - \mu_{3,6}''\mu_{4,5}'', \\
g_{E_1+Q_{3,1}} &= b(c-d)\eta_1''\lambda_3'' + (c-1)\eta_3''\lambda_1'' - \mu_{2,4}''\mu_{5,6}'' + \mu_{2,6}''\mu_{4,5}'', \\
g_{E_1+Q_{3,2}} &= \gamma_1\eta_1''\lambda_3'' + (a-c)\eta_3''\lambda_1'' - \mu_{2,5}''\mu_{4,6}'' - \mu_{2,6}''\mu_{4,5}'', \\
g_{E_1+Q_{4,1}} &= bc\eta_1''\lambda_4'' + (bc-b-c+1)\eta_4''\lambda_1'' - \mu_{2,3}''\mu_{5,6}'' + \mu_{2,6}''\mu_{3,5}'', \\
g_{E_1+Q_{4,2}} &= (ad-bc)\eta_1''\lambda_4'' + \gamma_2\eta_4''\lambda_1'' + \mu_{2,5}''\mu_{3,6}'' - \mu_{2,6}''\mu_{3,5}'', \\
g_{E_1+Q_{5,1}} &= (d-c)\eta_1''\lambda_5'' + \gamma_2\eta_5''\lambda_1'' - \mu_{2,4}''\mu_{3,6}'' + \mu_{2,6}''\mu_{3,4}'', \\
g_{E_1+Q_{5,2}} &= c\eta_1''\lambda_5'' + (a-c)(1-b)\eta_5''\lambda_1'' - \mu_{2,3}''\mu_{4,6}'' - \mu_{2,6}''\mu_{3,4}'', \\
g_{E_1+Q_{6,1}} &= (b-a)\eta_1''\lambda_6'' + \gamma_2\eta_6''\lambda_1'' + \mu_{2,4}''\mu_{3,5}'' - \mu_{2,5}''\mu_{3,4}'', \\
g_{E_1+Q_{6,2}} &= a\eta_1''\lambda_6'' + (c-a)(d-1)\eta_6''\lambda_1'' - \mu_{2,3}''\mu_{4,5}'' + \mu_{2,5}''\mu_{3,4}''.
\end{aligned}$$

As explained in the previous section, we may assume  $\eta_1'' = \dots = \eta_6'' = 1$ .

Recall the discussion before the definition (2) of  $F_{D,j}$  and before Lemma 10. If we had chosen  $\xi_4 = \eta_i\lambda_1$  and  $\xi_5 = \eta_1\lambda_i$  when writing down the equations  $F_{E_1+Q_{i,j}}$  in [Der06a, Section 3], then the resulting  $g_{E_1+Q_{i,2}}$  would give  $\lambda_i''$  directly as a quadratic expression in terms of  $\mu_{j,k}''$ . Furthermore, each of the five  $g_{E_1+Q_{i,1}}$  would express  $\lambda_1''$  as a quadratic equation in  $\mu_{j,k}''$ . Of course, we get the same result by solving the equivalent system of equations  $g_{E_1+Q_{i,j}}$  as listed above.

The equations  $g_{E_1+m_{1,i},j}$  for  $i \in \{3, \dots, 6\}$  and  $g_{E_1+Q_{2,j}}$  allow us to express the variables  $\mu_{1,i}'', \lambda_2''$  in  $\Xi_{1,0}'$  and  $\mu_{2,i}'', \lambda_1''$  in  $\Xi_{0,1}'$  in terms of the six variables  $\mu_{3,4}'', \dots, \mu_{5,6}'' \in \Xi_{0,0}'$ . (As we have set the remaining elements  $\eta_3'', \dots, \eta_6''$  of  $\Xi_{0,0}'$  to the value 1, they do not occur in these expressions.)

With  $\gamma_3 := d(a-c)(1-b) - c(b-d)(1-a)$ , we obtain:

$$\begin{aligned}
\mu_{1,3}'' &= \mu_{3,4}'' + \mu_{3,5}'' + \mu_{3,6}'', \\
\mu_{2,3}'' &= -\mu_{3,4}'' - a\mu_{3,5}'' - c\mu_{3,6}'', \\
\mu_{1,4}'' &= \mu_{3,4}'' + (b-1)\mu_{4,5}'' + (1-d)\mu_{4,6}'', \\
\mu_{2,4}'' &= \mu_{3,4}'' + (b-a)\mu_{4,5}'' + (c-d)\mu_{4,6}'', \\
\mu_{1,5}'' &= 1/b\mu_{3,5}'' + (1-b)/b\mu_{4,5}'' + (d-b)/b\mu_{5,6}'', \\
\mu_{2,5}'' &= a/b\mu_{3,5}'' + (a-b)/b\mu_{4,5}'' + \gamma_1/b\mu_{5,6}'', \\
\mu_{1,6}'' &= 1/d\mu_{3,6}'' + (d-1)/d\mu_{4,6}'' + (b-d)/d\mu_{5,6}'', \\
\mu_{2,6}'' &= c/d\mu_{3,6}'' + (d-c)/d\mu_{4,6}'' - \gamma_1/d\mu_{5,6}'', \\
\lambda_1'' &= -\gamma_1/\gamma_3\mu_{3,4}''\mu_{5,6}'' - a(d-c)/\gamma_3\mu_{3,5}''\mu_{4,6}'' - c(b-a)/\gamma_3\mu_{3,6}''\mu_{4,5}'', \\
\lambda_2'' &= (b-d)/\gamma_3\mu_{3,4}''\mu_{5,6}'' + (1-d)/\gamma_3\mu_{3,5}''\mu_{4,6}'' + (1-b)/\gamma_3\mu_{3,6}''\mu_{4,5}''.
\end{aligned}$$

For  $E \in \{m_{1,2}, Q_3, \dots, Q_6\}$ , we consider the remaining equations  $g_{E_1+E,i}$ . We can use  $g_{E_1+E,2}$  and substitution of our previous results in order to

express  $\xi''(E) \in \Xi''_{1,1}$  in terms of  $\Xi''_{0,0}$ :

$$\begin{aligned}
\mu''_{1,2} &= \mu''_{3,4} - a/b\mu''_{3,5} - c/d\mu''_{3,6} + (a-b)(b-1)/b\mu''_{4,5} \\
&\quad + (d-c)(d-1)/d\mu''_{4,6} + (b-d)\gamma_1/(bd)\mu''_{5,6}, \\
\lambda''_3 &= (a-c)/\gamma_3\mu''_{3,4}\mu''_{5,6} + a(1-c)/(b\gamma_3)\mu''_{3,5}\mu''_{4,6} + c(1-a)/(d\gamma_3)\mu''_{3,6}\mu''_{4,5} \\
&\quad + 1/(bd)\mu''_{4,5}\mu''_{4,6} - 1/d\mu''_{4,5}\mu''_{5,6} + 1/b\mu''_{4,6}\mu''_{5,6}, \\
\lambda''_4 &= \gamma_2/\gamma_3\mu''_{3,4}\mu''_{5,6} - 1/(bd)\mu''_{3,5}\mu''_{3,6} + (1-d)(c-d)(a-1)/(d\gamma_3)\mu''_{3,5}\mu''_{4,6} \\
&\quad - 1/d\mu''_{3,5}\mu''_{5,6} + (1-c)(b-1)(a-b)/(b\gamma_3)\mu''_{3,6}\mu''_{4,5} - 1/b\mu''_{3,6}\mu''_{5,6}, \\
\lambda''_5 &= 1/d\mu''_{3,4}\mu''_{3,6} - 1/d\mu''_{3,4}\mu''_{4,6} + (b-d)(1-a)\gamma_1/(d\gamma_3)\mu''_{3,4}\mu''_{5,6} \\
&\quad + a\gamma_2/\gamma_3\mu''_{3,5}\mu''_{4,6} + (b-1)(a-b)(a-c)/\gamma_3\mu''_{3,6}\mu''_{4,5} - \mu''_{3,6}\mu''_{4,6}, \\
\lambda''_6 &= -1/b\mu''_{3,4}\mu''_{3,5} - 1/b\mu''_{3,4}\mu''_{4,5} + (1-c)(b-d)\gamma_1/(b\gamma_3)\mu''_{3,4}\mu''_{5,6} \\
&\quad - \mu''_{3,5}\mu''_{4,5} + (d-1)(c-d)(a-c)/\gamma_3\mu''_{3,5}\mu''_{4,6} + c\gamma_2/\gamma_3\mu''_{3,6}\mu''_{4,5}
\end{aligned}$$

Finally, we check by substituting and expanding that the five further conditions  $g_{E_1+E,1}$  are trivial.

Using the restrictions on  $a, b, c, d$  imposed by the fact that  $p_1, \dots, p_6$  are in general position (e.g.,  $a$  must be different from  $b$  and  $c$ , and all are neither 0 nor 1), we see that  $\mu''_{1,2}, \dots, \mu''_{2,6}, \lambda''_1, \dots, \lambda''_6$  are non-zero polynomials in  $\mu''_{3,4}, \dots, \mu''_{5,6}$ . Therefore, for an open subset of the  $N_4 + 2 = 12$  parameters  $\eta''_1, \dots, \eta''_6, \mu''_{3,4}, \dots, \mu''_{5,6}$ , all rescaling factors are non-zero, resulting in good rescalings.

## 6. DEGREE 2

For the proof of Lemma 11 for  $r = 7$ , we proceed as in the case  $r = 6$  and assume  $\eta''_1 = \dots = \eta''_7 = 1$ .

Let  $D := D_i^{(1)} \in \mathcal{M}_7$  for  $i \in \{3, \dots, 7\}$ . We can arrange  $F_{D,1}, \dots, F_{D,4}$  in such a way that  $g_{D,1}$  and  $g_{D,2}$  express  $\mu''_{1,i}$  and  $\mu''_{2,i}$  in terms of  $\mu''_{i,j}$  for  $j \in \{3, \dots, 7\} \setminus \{i\}$  (see the discussion before Lemma 10). Similarly, for  $i \in \{3, \dots, 7\}$  and  $D := D_{1,2,i}^{(2)} \in \mathcal{M}_7$ , we can arrange  $g_{D,1}$  and  $g_{D,2}$  such that they express  $\nu''_{1,i}$  and  $\nu''_{2,i}$  linearly in  $\nu''_{1,2}$  and of degree 2 in  $\mu''_{3,4}, \dots, \mu''_{6,7}$ . This expresses all variables in  $\Xi''_{0,1} \cup \Xi''_{1,0}$  in terms of  $\Xi''_{0,0}$ .

Substituting this into an appropriately arranged  $g_{D,2}$  for  $D := D_2^{(1)} \in \mathcal{M}_7$  gives  $\mu''_{1,2} \in \Xi''_{1,1}$  in terms of  $\mu''_{3,4}, \dots, \mu''_{6,7}$ , and we check that  $g_{D,1}$  becomes trivial.

Now, let  $D := D_{1,i,j}^{(2)} \in \mathcal{M}_7$  for  $i < j \in \{3, \dots, 7\}$ . We arrange  $g_{D,1}$  and  $g_{D,2}$  such that they express  $\nu''_{1,i}$  respectively  $\nu''_{i,j}$  in terms of  $\nu''_{1,j}$  and expressions of degree 2 in  $\mu''_{k,l}$ . Using our previous findings, the first expression turns out trivial, and the second one gives  $\nu''_{i,j} \in \Xi''_{1,1}$  in terms of  $\nu''_{1,2}, \mu''_{3,4}, \dots, \mu''_{6,7} \in \Xi''_{0,0}$ .

Finally, let  $D \in \{D_{1,2}^{(3)}, \dots, D_{1,7}^{(3)}, D_{2,1}^{(3)}\} \subset \mathcal{M}_7$ . We arrange  $g_{D,1}$  and  $g_{D,2}$  such that the first one is an expression in  $\mu''_{j,k}$  and  $\nu''_{j,k}$  which becomes trivial. The second one expresses  $\lambda''_i$  in terms of  $\mu''_{j,k}$  and  $\nu''_{j,k}$ , and we substitute again our previous results.

This completes the proof of Lemma 11 and thus Theorem 1. In total, we obtain good rescalings for an open subset of a system  $N_5 + 2 = 18$  parameters

$$\eta''_1, \dots, \eta''_7, \quad \mu''_{3,4}, \dots, \mu''_{6,7}, \quad \nu''_{1,2}.$$

Since it is straightforward to determine the exact expressions for the remaining 38 rescaling factors in terms of these parameters, and since the expressions are rather long, we choose not to list them here.

*Remark 12.* In principle, it would be possible to consider the conditions  $g_{D,i}$  for all (1)-rulings without reducing to the subset  $\mathcal{M}_r$  as we did in Section 3. While this is doable in degree 3 with some software help (**Magma**), especially the expressions corresponding to the (1)-ruling  $D_i^{(5)}$  in degree 2 seem to be out of reach for direct computations. Furthermore, we would have to embed the relations  $v_i^j$  corresponding to the (2)-ruling  $-K_7$ , which causes further complications.

*Remark 13.* For  $r \in \{5, 6, 7\}$ , there is a  $(N_{r-2} - r + 1)$ -parameter family of images of  $\mathbb{A}(S_r)$  under good embeddings in  $H_r$  by Remark 2. The dimension of  $\mathbb{A}(S_r)$  is  $r + 3$ , and there is a  $(2 \cdot (r - 4))$ -parameter family of smooth Del Pezzo surfaces  $S_r$  of degree  $9 - r$ . The dimension of  $H_r$  is  $N_{r-1} + 1$ .

In fact, [Pop01, Section 6.3] shows that the closure of the union of all these images for all Del Pezzo surfaces of degree 4 equals  $H_5$ .

For  $r = 6$ , by comparing the dimensions and numbers of parameters, a similar result seems possible. However, for  $r = 7$ , we have

$$(N_{r-2} - r + 1) + (r + 3) + 2 \cdot (r - 4) = 26,$$

while  $H_7$  has dimension  $N_{r-1} + 1 = 28$ . Consequently, the closure of the union of the corresponding images, over all Del Pezzo surfaces of degree 2, under all good embeddings cannot be  $H_7$  for dimension reasons.

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